

ON DIBARIC AND EVOLUTION ALGEBRAS

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ABSTRACT. We find conditions on ideals of an algebra under which the algebra is dibaric. Dibaric algebras have not non-zero homomorphisms to the set of the real numbers. We introduce a concept of bq-homomorphism (which is given by two linear maps f, g of the algebra to the set of the real numbers) and show that an algebra is dibaric if and only if it admits a non-zero bq-homomorphism. Using the pair (f, g) we define conservative algebras and establish criteria for a dibaric algebra to be conservative. Moreover, the notions of a Bernstein algebra and an algebra induced by a linear operator are introduced and relations between these algebras are studied. For dibaric algebras we describe a dibaric algebra homomorphism and study their properties by bq-homomorphisms of the dibaric algebras. We apply the results to the (dibaric) evolution algebra of a bisexual population. For this dibaric algebra we describe all possible bq-homomorphisms and find conditions under which the algebra of a bisexual population is induced by a linear operator. Moreover, some properties of dibaric algebra homomorphisms of such algebras are studied.

1. INTRODUCTION

There exist several classes of non-associative algebras (baric, evolution, Bernstein, train, stochastic, etc.), whose investigation has provided a number of significant contributions to theoretical population genetics. Such classes have been defined different times by several authors, and all algebras belonging to these classes are generally called genetic. Etherington introduced the formal language of abstract algebra to the study of the genetics in his series of seminal papers [6–8]. In recent years many authors have tried to investigate the difficult problem of classification of these algebras. The most comprehensive references for the mathematical research done in this area are [13, 15–17].

In [16] a new type of evolution algebra is introduced. This algebra also describes some evolution laws of genetics and it is an algebra E over a field K with a countable natural basis e_1, e_2, \dots and multiplication given by $e_i e_i = \sum_j a_{ij} e_j$, $e_i e_j = 0$ if $i \neq j$. Therefore, $e_i e_i$ is viewed as “self-reproduction”.

In book [13] an evolution algebra \mathcal{A} associated to the free population is introduced and using this non-associative algebra many results are obtained in explicit form, e.g. the explicit description of stationary quadratic operators, and the explicit solutions of a nonlinear evolutionary equation in the absence of selection, as well as general theorems on convergence to equilibrium in the presence of selection. Moreover, this book deals with baric algebras which have non-zero homomorphism to the set of real numbers. The theory of baric algebras is well developed (see for example [1–3, 5–10, 13, 14]).

But there exist many non baric algebras, for example an evolution algebra of a bisexual population. An algebra is called *dibarcic* if it has a non trivial homomorphism onto the sex differentiation algebra. The study of dibarcic algebras has as motivation the algebras coming from genetic models in bisexual populations with sex linked genetic inheritance. First, Etherington [6], introduced the idea of treating the male and female components of a population separately and next Holgate [11] formalized this concept with the introduction of the sex differentiation algebra and dibarcic algebras. Following the modern notation of [17], we give Holgate's definitions bellow. See also the survey [15] for more information.

In [5] basic properties of dibarcic algebras are given. The authors define the union of two dibarcic algebras, following the same lines that were used by Costa and Guzzo [4] for baric algebras having an idempotent element of weight 1, and also the notion of indecomposable algebra and the results obtained for baric algebras are generalized for dibarcic algebras. It is proved that the decomposition of a dibarcic algebra as the union of indecomposable dibarcic algebras is unique, assuming that the dibarcic algebra satisfies both ascending and descending chain conditions, and that it possesses a semiprincipal idempotent element.

In this paper we develop the theory of dibarcic algebras by applying some results to an evolution algebra of a bisexual population defined using inheritance coefficients of the population.

The paper is organized as follows. In Section 2 we give the evolution operator and the algebra of a bisexual population. Section 3 contains some general properties of dibarcic, Bernstein and conservative algebras. Section 4 is devoted to some properties of the evolution algebras of a bisexual population.

2. PRELIMINARIES

2.1. Evolution operator of a BP. In this subsection, following [13], we describe the evolution operator of a bisexual population (BP). Assuming that the population is bisexual we suppose that the set of females can be partitioned into finitely many different types indexed by $\{1, 2, \dots, n\}$ and, similarly, that the male types are indexed by $\{1, 2, \dots, \nu\}$. The number $n + \nu$ is called the *dimension of the population*. The population is described by its state vector (x, y) in $S^{n-1} \times S^{\nu-1}$, the product of two unit simplexes in \mathbb{R}^n and \mathbb{R}^ν respectively. Vectors x and y are the probability distributions of the females and males over the possible types:

$$x_i \geq 0, \sum_{i=1}^n x_i = 1; \quad y_i \geq 0, \sum_{i=1}^{\nu} y_i = 1.$$

Denote $S = S^{n-1} \times S^{\nu-1}$. We call the partition into types hereditary if for each possible state $z = (x, y) \in S$ describing the current generation, the state $z' = (x', y') \in S$ is uniquely defined describing the next generation. This means that the association $z \mapsto z'$ defines a map $V: S \rightarrow S$ called *the evolution operator*.

For any point $z^{(0)} \in S$ the sequence $z^{(t)} = V(z^{(t-1)}), t = 1, 2, \dots$ is called the *trajectory* of $z^{(0)}$.

Let $P_{ik,j}^{(f)}$ and $P_{ik,l}^{(m)}$ be the inheritance coefficients defined as the probability that a female offspring is type j and, respectively, that a male offspring is of type l , when the parental pair is ik ($i, j = 1, \dots, n$; and $k, l = 1, \dots, \nu$). We have

$$P_{ik,j}^{(f)} \geq 0, \quad \sum_{j=1}^n P_{ik,j}^{(f)} = 1; \quad P_{ik,l}^{(m)} \geq 0, \quad \sum_{l=1}^{\nu} P_{ik,l}^{(m)} = 1. \quad (2.1)$$

Let $z' = (x', y')$ be the state of the offspring population at the birth stage. This is obtained from the inheritance coefficients as

$$x'_j = \sum_{i,k=1}^{n,\nu} P_{ik,j}^{(f)} x_i y_k; \quad y'_l = \sum_{i,k=1}^{n,\nu} P_{ik,l}^{(m)} x_i y_k. \quad (2.2)$$

We see from (2.2) that for a BP the evolution operator is a quadratic mapping of S into itself.

2.2. An algebra of the bisexual population. In this subsection following [12] we give an algebra structure on the vector space $\mathbb{R}^{n+\nu}$ which is closely related to the map (2.2).

Consider $\{e_1, \dots, e_{n+\nu}\}$ the canonical basis on $\mathbb{R}^{n+\nu}$ and divide the basis as $e_i^{(f)} = e_i$, $i = 1, \dots, n$ and $e_i^{(m)} = e_{n+i}$, $i = 1, \dots, \nu$.

Now introduce on $\mathbb{R}^{n+\nu}$ a multiplication defined by

$$\begin{aligned} e_i^{(f)} e_k^{(m)} &= e_k^{(m)} e_i^{(f)} = \frac{1}{2} \left(\sum_{j=1}^n P_{ik,j}^{(f)} e_j^{(f)} + \sum_{l=1}^{\nu} P_{ik,l}^{(m)} e_l^{(m)} \right); \\ e_i^{(f)} e_j^{(f)} &= 0 & i, j = 1, \dots, n; \\ e_k^{(m)} e_l^{(m)} &= 0 & k, l = 1, \dots, \nu. \end{aligned} \quad (2.3)$$

Thus we identify the coefficients of bisexual inheritance as the structure constants of an algebra, i.e. a bilinear mapping of $\mathbb{R}^{n+\nu} \times \mathbb{R}^{n+\nu}$ to $\mathbb{R}^{n+\nu}$.

The general formula for the multiplication is the extension of (2.3) by bilinearity, i.e. for $z, t \in \mathbb{R}^{n+\nu}$,

$$z = (x, y) = \sum_{i=1}^n x_i e_i^{(f)} + \sum_{j=1}^{\nu} y_j e_j^{(m)}, \quad t = (u, v) = \sum_{i=1}^n u_i e_i^{(f)} + \sum_{j=1}^{\nu} v_j e_j^{(m)}$$

using (2.3), we obtain

$$\begin{aligned} zt &= \frac{1}{2} \sum_{k=1}^n \left(\sum_{i=1}^n \sum_{j=1}^{\nu} P_{ij,k}^{(f)} (x_i v_j + u_i y_j) \right) e_k^{(f)} \\ &\quad + \frac{1}{2} \sum_{l=1}^{\nu} \left(\sum_{i=1}^n \sum_{j=1}^{\nu} P_{ij,l}^{(m)} (x_i v_j + u_i y_j) \right) e_l^{(m)} \end{aligned} \quad (2.4)$$

From (2.4) and using (2.2), in the particular case that $z = t$, i.e. $x = u$ and $y = v$, we obtain

$$\begin{aligned} zz = z^2 &= \sum_{k=1}^n \left(\sum_{i=1}^n \sum_{j=1}^\nu P_{ij,k}^{(f)} x_i y_j \right) e_k^{(f)} \\ &+ \sum_{l=1}^\nu \left(\sum_{i=1}^n \sum_{j=1}^\nu P_{ij,l}^{(m)} x_i y_j \right) e_l^{(m)} = V(z) \end{aligned} \quad (2.5)$$

for any $z \in S$.

This algebraic interpretation is very useful. For example, a BP state $z = (x, y)$ is an equilibrium (fixed point, $V(z) = z$) precisely when z is an idempotent element of the set S .

If we write $z^{[t]}$ for the power $\underbrace{(\cdots (z^2)^2 \cdots)}_{t\text{-times}}$, with $z^{[0]} \equiv z$, then the trajectory with initial state z is $V^t(z) = z^{[t]}$.

The algebra $\mathcal{B} = \mathcal{B}_V$ generated by the evolution operator V (see (2.2)) is called the *evolution algebra of the bisexual population* (EABP).

The following theorem gives basic properties of the EABP.

Theorem 2.3 ([12]).

- (1) *Algebra \mathcal{B} is not associative, in general.*
- (2) *Algebra \mathcal{B} is commutative, flexible.*
- (3) *\mathcal{B} is not power-associative, in general.*

A *character* for an algebra \mathbf{A} is a non-zero multiplicative linear form on \mathbf{A} , that is, a non-zero algebra homomorphism from \mathbf{A} to \mathbb{R} [13]. A pair (\mathcal{A}, σ) consisting of an algebra \mathcal{A} and a character σ on \mathcal{A} is called a *baric algebra*. In [13] for the EA of a free population it is proven that there is a character $\sigma(x) = \sum_i x_i$, therefore that algebra is baric. But in [12] it is proven that the EABP, i.e. \mathcal{B} is not baric.

3. DIBARIC ALGEBRAS

As usual, the algebras considered in mathematical biology are not baric. In particular, the algebra \mathcal{B} is not a baric algebra. To overcome such complication, Etherington [8] for a zygotic algebra of sex linked inheritance introduced the idea of treating the male and female components of a population separately. In [11] Holgate formalized this concept by introducing sex differentiation algebras and a generalization of baric algebras called dibaric algebras. In this section we shall introduce a concept of bi-quasi-homomorphism (in short: bq-homomorphism) and establish criteria for an algebra to be dibaric algebra.

Definition 3.1 ([15, 17]). Let $\mathfrak{A} = \langle w, m \rangle_{\mathbb{R}}$ denote a two dimensional commutative algebra over \mathbb{R} with multiplicative table

$$w^2 = m^2 = 0, \quad wm = \frac{1}{2}(w + m).$$

Then \mathfrak{A} is called the *sex differentiation algebra*.

As usual, a subalgebra \mathbf{B} of an algebra \mathbf{A} is a subspace which is closed under multiplication. A subspace \mathbf{B} is an ideal if it is closed under multiplication by all elements of \mathbf{A} . For example, the square of the algebra:

$$\mathbf{A}^2 = \text{span}\{zt : z, t \in \mathbf{A}\}$$

is an ideal.

It is clear that $\mathfrak{A}^2 = \langle w + m \rangle_{\mathbb{R}}$ is an ideal of \mathfrak{A} which is isomorphic to the field \mathbb{R} . Hence the algebra \mathfrak{A}^2 is a baric algebra. Now we can define Holgate's generalization of a baric algebra.

Definition 3.2 ([15]). *An algebra is called dibaric if it admits a homomorphism onto the sex differentiation algebra \mathfrak{A} .*

Proposition 3.3. *Let \mathbf{A} be a commutative dibaric algebra over the field \mathbb{R} . Then there is an ideal $I \triangleleft \mathbf{A}$ with $\text{codim}(I) = 2$ such that $\mathbf{A}^2 \not\subseteq I$.*

Proof. For a dibaric algebra we have a non-zero homomorphism $\varphi: \mathbf{A} \rightarrow \mathfrak{A}$. Put $I = \ker \varphi$, then since φ is onto we have $\text{codim}(I) = 2$. Therefore \mathbf{A} can be represented as $\mathbf{A} = \mathbb{R}e_1 + \mathbb{R}e_2 + I$, and we can take $\varphi(e_1) = w$ and $\varphi(e_2) = m$. Then $\varphi(e_1e_2) = \frac{1}{2}(w + m) \neq 0$, consequently $e_1e_2 \in \mathbf{A}^2$, but $e_1e_2 \notin I$. Hence $\mathbf{A}^2 \not\subseteq I$. \square

Theorem 3.4. *Let \mathbf{A} be a commutative algebra over the field \mathbb{R} , which satisfies the following conditions:*

- (1) *There is an ideal $I \triangleleft \mathbf{A}$ with $\text{codim}(I) = 2$;*
- (2) *There exists $\bar{e} \in \mathbf{A}/I$ such that $\mathbf{A}^2 = \mathbb{R}\bar{e} + I$ and $\bar{e}^2 - \bar{e} \in I$;*
- (3) *There exists $x \in \mathbf{A}$ such that $x^2 + \bar{e} \in I$.*

Then the algebra \mathbf{A} is a dibaric algebra.

Proof. Consider the natural homomorphism $\varphi: \mathbf{A} \rightarrow \mathbf{A}/I$ which is onto. We shall prove that under conditions (1)-(3) $\varphi(\mathbf{A}) = \mathbf{A}/I$ is isomorphic to \mathfrak{A} . Here we shall use the fact that \mathfrak{A} is isomorphic to an algebra $\langle p, q \rangle$ with multiplication $p^2 = p$, $q^2 = -p$, $pq = 0$, where $p = m + w$, $q = m - w$. From (1) we get $\dim(\mathbf{A}/I) = 2$, i.e. $\mathbf{A}/I = \langle \bar{p}, \bar{q} \rangle$.

From (2) we obtain $(\mathbf{A}/I)^2 = \langle \bar{p} \rangle$, where $\bar{p}^2 = \bar{p}$. Therefore, the algebra \mathbf{A}/I has the following table of multiplication:

$$\bar{p}^2 = \bar{p}, \quad \bar{p}\bar{q} = \alpha_1\bar{p}, \quad \bar{q}^2 = \alpha_2\bar{p}.$$

The change $\bar{q}' = \bar{q} - \alpha_1\bar{p}$ allows to take $\alpha_1 = 0$. Hence, we consider the table of multiplication:

$$\bar{p}^2 = \bar{p}, \quad \bar{p}\bar{q}' = 0, \quad \bar{q}'^2 = \alpha_2\bar{p}.$$

Now consider the following possible cases:

Case $\alpha_2 = 0$: In this case, the equation $\bar{x}^2 = -\bar{p}$ is equivalent to $\alpha^2\bar{p} = -\bar{p}$, which has not solution $\alpha \in \mathbb{R}$, i.e. the condition (3) is not satisfied. Clearly, the corresponding algebra is not isomorphic to \mathfrak{A} .

Case $\alpha_2 < 0$: In this case, the change $\bar{q}' = \frac{\bar{q}}{\sqrt{|\alpha_2|}}$ allows to put $\alpha_2 = -1$. Consequently, \mathbf{A}/I is isomorphic to \mathfrak{A} .

Case $\alpha_2 > 0$: In this case, the change $\bar{q}' = \frac{\bar{q}}{\sqrt{\alpha_2}}$ allows to put $\alpha_2 = 1$. The equation $\bar{x}^2 = -\bar{p}$ is equivalent to $(\alpha^2 + \beta^2)\bar{p} = -\bar{p}$, which has not solution $\alpha, \beta \in \mathbb{R}$, i.e. the condition (3) is not satisfied. Clearly, the corresponding algebra is not isomorphic to \mathfrak{A} .

□

Remark 3.5.

1. From (2) we get $\mathbf{A}^2 \not\subseteq I$, but the converse is not true in general.
2. If \mathbf{A}/I is isomorphic to $\langle \bar{p}, \bar{q} \rangle$ with $\bar{p}^2 = 0$, $\bar{p}\bar{q} = \bar{p}$, $\bar{q}^2 = \bar{p}$, then the condition (3) of Theorem 3.4 is satisfied, but the condition (2) is not satisfied. Clearly, the corresponding algebra \mathbf{A}/I is not isomorphic to \mathfrak{A} .
3. If \mathbf{A}/I is isomorphic to $\langle \bar{p}, \bar{q} \rangle$ with $\bar{p}^2 = \bar{p}$, $\bar{p}\bar{q} = 0$, $\bar{q}^2 = \bar{p}$, then the condition (2) of Theorem 3.4 is satisfied, but the condition (3) is not satisfied. It is easy to see that the corresponding algebra \mathbf{A}/I is not isomorphic to \mathfrak{A} .

Definition 3.6. For a given algebra \mathbf{A} , a pair (f, g) , of linear forms $f: \mathbf{A} \rightarrow \mathbb{R}$, $g: \mathbf{A} \rightarrow \mathbb{R}$ is called bq-homomorphism if

$$f(xy) = g(xy) = \frac{f(x)g(y) + f(y)g(x)}{2} \quad \text{for any } x, y \in \mathbf{A}. \quad (3.1)$$

Note that if $f = g$ then the condition (3.1) implies that f is a homomorphism.

A bq-homomorphism (f, g) is called non-zero if $f(z)g(z) \neq 0$, i.e. both f and g are non-zero.

Theorem 3.7. An algebra \mathbf{A} is dibaric if and only if there is a non-zero bq-homomorphism.

Proof. Assume \mathbf{A} admits a non-zero bq-homomorphism (f, g) . Consider mapping $\varphi: \mathbf{A} \rightarrow \mathfrak{A}$ defined by

$$\varphi(x) = f(x)w + g(x)m.$$

For $x, y \in \mathbf{A}$, we have

$$\varphi(xy) = f(xy)w + g(xy)m = \frac{f(x)g(y) + f(y)g(x)}{2}(w + m).$$

Using $w^2 = m^2 = 0$, $wm = \frac{1}{2}(w + m)$ we get

$$\varphi(x)\varphi(y) = (f(x)w + g(x)m)(f(y)w + g(y)m) = (f(x)g(y) + f(y)g(x))wm = \varphi(xy),$$

i.e. φ is a homomorphism. For arbitrary $u = \alpha w + \beta m \in \mathfrak{A}$ it is easy to see that $\varphi(x) = u$ if $f(x) = \alpha$ and $g(x) = \beta$. Therefore φ is onto. Hence \mathbf{A} is dibaric.

Conversely, if \mathbf{A} is dibaric then there is homomorphism φ from \mathbf{A} onto \mathfrak{A} , which has the form $\varphi(x) = W(x)w + M(x)m$. Since φ is onto, $W \neq 0$ and $M \neq 0$. We have

$$\varphi(xy) = W(xy)w + M(xy)m = \varphi(x)\varphi(y) = \frac{W(x)M(y) + W(y)M(x)}{2}(w + m),$$

which implies that

$$W(xy) = M(xy) = \frac{W(x)M(y) + W(y)M(x)}{2}.$$

Thus for non-zero bq-homomorphism we can take $(f(x), g(x)) = (W(x), M(x))$. \square

The following proposition is useful.

Proposition 3.8 ([11]). *If an algebra \mathbf{A} is dibaric, then \mathbf{A}^2 is baric.*

In [12] it is proven that an EABP \mathcal{B} is dibaric, hence the subalgebra \mathcal{B}^2 is a baric algebra.

A pair $(\mathbf{A}, (f, g))$ consisting of an algebra \mathbf{A} and a non-zero bq-homomorphism (f, g) denotes a dibaric algebra.

A linear form, F , on a dibaric algebra, $(\mathbf{A}, (f, g))$, is called *f-invariant linear form* (resp. *g-invariant linear form*) if it satisfies the equality

$$F(x^2) = f(x)F(x), \quad (\text{resp. } F(x^2) = g(x)F(x)) \quad x \in \mathbf{A}. \quad (3.2)$$

or equivalently:

$$F(xy) = \frac{f(x)F(y) + f(y)F(x)}{2}, \quad \left(\text{resp. } F(xy) = \frac{g(x)F(y) + g(y)F(x)}{2} \right) \quad x, y \in \mathbf{A}. \quad (3.3)$$

Invariant forms define conservation laws for the dynamical system. The gene conservation laws are examples (see [13] for details).

Denote by J_f (resp. J_g) the set of all *f*-invariant (resp. *g*-invariant) linear forms of \mathbf{A} . The set J_f and J_g are subspaces of the dual space \mathbf{A}^* . Clearly, $F(x) \equiv 0$ is an element of $J_f \cap J_g$. Moreover, $g \in J_f$ and $f \in J_g$. Hence $1 \leq \dim J_f, \dim J_g \leq \dim \mathbf{A}$.

Remark 3.9. *Note that any baric algebra (\mathcal{A}, σ) is dibaric with $f = g = \sigma$. But a dibaric algebra is not baric in general. For example, an EABP \mathcal{B} is dibaric but not baric.*

Lemma 3.10. *If $(\mathbf{A}, (f, g))$ is a dibaric (not baric) algebra then $J_f \cap J_g = \{0\}$.*

Proof. Assume $F \in J_f \cap J_g$ then from (3.3) we get

$$(f(x) - g(x))F(y) + (f(y) - g(y))F(x) = 0, \quad (3.4)$$

and for $x = y$ we have

$$(f(x) - g(x))F(x) = 0. \quad (3.5)$$

Now if $f(x) - g(x) = 0$ then we take y such that $f(y) - g(y) \neq 0$ then from (3.4) it follows that $F(x) = 0$. If $f(x) - g(x) \neq 0$ then from (3.5) it follows that $F(x) = 0$. Thus $F \equiv 0$. \square

Since in the definition of a bq-homomorphism (f, g) , the functions f and g play a “symmetric” role, we consider only J_f in the sequel of this section.

Denote

$$J_f^\perp = \{x : F(x) = 0, \text{ for all } F \in J_f\},$$

$$\text{ann } \mathbf{A} = \{y \in \mathbf{A} : yx = 0, \text{ for all } x \in \mathbf{A}\}.$$

Lemma 3.11. $\text{ann } \mathbf{A} \subseteq J_f^\perp$.

Proof. If $x \in \text{ann } \mathbf{A}$ then using equation (3.3) we obtain $f(x)F(y) + f(y)F(x) = 0$, $y \in \mathbf{A}$. Hence if $f(x) = 0$ then we take y such that $f(y) \neq 0$, consequently $F(x) = 0$. If $f(x) \neq 0$ then from $0 = F(x^2) = f(x)F(x)$ we get $F(x) = 0$ for any $F \in J_f$, i.e. $x \in J_f^\perp$. \square

Definition 3.12. The algebra \mathbf{A} is called conservative if

$$J_f^\perp = \text{ann } \mathbf{A}. \quad (3.6)$$

Theorem 3.13. In order to \mathbf{A} being conservative it is necessary and sufficient that the product xy depends only upon the values of the invariant forms on x and y . That is

$$x \equiv x' \text{ and } y \equiv y' \pmod{J_f^\perp} \text{ imply } xy = x'y'. \quad (3.7)$$

Proof. Necessariness. Let \mathbf{A} be conservative, then $J_f^\perp = \text{ann } \mathbf{A}$. Consider x, y, x', y' with $x - x', y - y' \in \text{ann } \mathbf{A}$. Consequently, $(x - x')z = 0, (y - y')z = 0, \forall z \in \mathbf{A}$. From $(x - x')(y - y') = 0$ and from these equations for $z = x', y'$ we get $x'y - x'y' = 0, xy' - x'y' = 0$, which implies $xy = x'y'$.

Sufficiency. Now assume that (3.7) is satisfied, we shall show that \mathbf{A} is a conservative algebra. Apply (3.7) with $y = y', x' = 0$. We get $x \in J_f^\perp$ which implies $xy = 0$. Hence, $J_f^\perp \subset \text{ann } \mathbf{A}$ which, together with Lemma 3.11, imply (3.6). \square

Corollary 3.14. Let $\{F_1, \dots, F_m\}$ be a basis of the subspace J_f of invariant forms for the algebra \mathbf{A} . The algebra \mathbf{A} is conservative if and only if there exist vectors $u_{ik} \in \mathbf{A}$ ($1 \leq i, k \leq m$) with $u_{ik} = u_{ki}$ so that the multiplication is given by the formula

$$xy = \sum_{i,k=1}^m F_i(x)F_k(y)u_{ik}, \quad x, y \in \mathbf{A}.$$

Proposition 3.15. An algebra \mathbf{A} is conservative if and only if

$$x^2y = f(x)xy, \quad x, y \in \mathbf{A}.$$

Proof. The proof is very similar to the proof of [13, Theorem 3.3.6]. \square

Define the following powers of $x \in \mathbf{A}$:

$$x^{[1]} = x^2, \quad x^{[k+1]} = (x^{[k]})^2, \quad k \geq 1.$$

Proposition 3.16. In a conservative algebra \mathbf{A} the following identity holds:

$$x^{[k]} = (f(x))^{2^k-2}x^2, \quad k \geq 1, \quad x \in \mathbf{A}. \quad (3.8)$$

Proof. Using (3.7) we get

$$(x^2)^2 = (f(x))^2x^2. \quad (3.9)$$

Now assume that formula (3.8) is true for $k-1$ and prove it for k :

$$x^{[k]} = (x^{[k-1]})^2 = \left((f(x))^{2^{k-1}-2}x^2\right)^2 = (f(x))^{2^k-4}(x^2)^2 = (f(x))^{2^k-2}x^2.$$

\square

The algebras \mathbf{A} in which (3.9) holds are called *stationary* or *Bernstein* algebras because of the connection between this class and the problem of Bernstein (see [13, Section 2.1 and Chapters 4, 5]). For an evolution algebra \mathcal{B} , condition (3.9) is equivalent to the stationary principle

$$V^2(z) = V(V(z)) = V(z),$$

where $V(z) = z^2$ is the evolution operator (2.2) on $S^{n-1} \times S^{\nu-1}$.

In [13] the concept of Bernstein algebra is introduced for baric algebras. But we are considering not baric algebras.

Remark 3.17. *Notice that each conservative algebra is Bernstein, but in [13, page 99] an example of a baric algebra (\mathcal{A}, σ) was constructed which is Bernstein but is not conservative. If we consider a dibaric algebra $(\mathbf{A}, (f, g))$, then the example (for $\sigma = f$) will be an example of a Bernstein algebra which is not conservative.*

Definition 3.18. *An algebra \mathbf{A} is called induced by a linear operator if there is a linear operator A in the vector space \mathbf{A} such that*

$$xy = \frac{f(x)A(y) + f(y)A(x)}{2}, \quad x, y \in \mathbf{A}. \quad (3.10)$$

Proposition 3.19. *The algebra \mathbf{A} induced by a linear operator A is Bernstein if and only if*

$$f(x)A(x) = f(A(x))A^2(x). \quad (3.11)$$

Proof. From (3.10) we get $x^2 = f(x)A(x)$; now if \mathbf{A} is Bernstein then from (3.9) we obtain

$$(f(x))^2 f(A(x))A^2(x) = (f(x))^2 x^2. \quad (3.12)$$

Denote

$$W = \{x \in \mathbf{A} : f(x) \neq 0\}.$$

Since the set W is a dense subset of \mathbf{A} , from (3.12) we get

$$f(A(x))A^2(x) = x^2 = f(x)A(x).$$

Assume now that the condition (3.11) is satisfied. Then

$$\begin{aligned} (x^2)^2 &= (f(x))^2 (A(x))^2 = (f(x))^2 f(A(x))A^2(x) \\ &= (f(x))^3 A(x) = (f(x))^2 f(x)A(x) = (f(x))^2 x^2. \end{aligned}$$

□

For an algebra \mathbf{A} denote by N the subspace of disappearing forms, i.e. the linear forms which vanish on the subalgebra \mathbf{A}^2 . We have

$$J_f \cap N = \{0\}.$$

Indeed, if $F(x^2) = f(x)F(x)$ ($F \in J_f$) and at the same time $F(x^2) = 0$ ($F \in N$) for all $x \in \mathbf{A}$ then F vanishes on the dense open subset of \mathbf{A} where $f(x) \neq 0$. Consequently, $F = 0$.

It follows that $\dim J_f + \dim N \leq \dim \mathbf{A}$.

Proposition 3.20. *If the algebra is induced by the linear operator A , then $N = (\text{Im } A)^\perp$.*

Proof. From (3.10) we get $xy = A\left(\frac{f(x)y + f(y)x}{2}\right)$. Consequently, $\mathbf{A}^2 \subset \text{Im } A$ and $(\text{Im } A)^\perp \subset N$. Conversely, for $F \in N$ we have $0 = F(x^2) = F(f(x)A(x)) = f(x)F(A(x))$. Since W is dense in \mathbf{A} we get $F(A(x)) = 0$ for all x . Hence $F \in (\text{Im } A)^\perp$. \square

Given dibaric algebras $(\mathbf{A}_1, (f_1, g_1))$ and $(\mathbf{A}_2, (f_2, g_2))$, a *dibaric algebra homomorphism* $h: (\mathbf{A}_1, (f_1, g_1)) \rightarrow (\mathbf{A}_2, (f_2, g_2))$ is an homomorphism $h: \mathbf{A}_1 \rightarrow \mathbf{A}_2$ such that $h^*f_2 = f_2h = f_1$. For example, the embedding of a dibaric subalgebra and the quotient map to a dibaric quotient algebra are dibaric homomorphisms. Clearly, the composition of dibaric homomorphisms is dibaric. A bijective dibaric homomorphism is called a *dibaric isomorphism*. The inverse of a dibaric isomorphism is dibaric because $h^*f_2 = f_1$ implies $f_2 = (h^{-1})^*f_1$.

The following proposition says that the definition of dibaric homomorphism does not depend on the choice of f and g .

Proposition 3.21. *For a dibaric algebra homomorphism $h: (\mathbf{A}_1, (f_1, g_1)) \rightarrow (\mathbf{A}_2, (f_2, g_2))$ we have $h^*f_2 = f_1$ if and only if $h^*g_2 = g_1$.*

Proof. Since f and g play symmetric role it suffices to prove that from $h^*f_2 = f_1$ follows $h^*g_2 = g_1$. So let $h^*f_2 = f_1$. Then we have

$$\begin{aligned} f_2(h(x^2)) &= f_2(h(x))g_2(h(x)) = f_1(x)g_2(h(x)) \\ f_1(x^2) &= f_1(x)g_1(x). \end{aligned}$$

Hence

$$f_1(x)(g_2(h(x)) - g_1(x)) = 0. \quad (3.13)$$

We also have

$$\begin{aligned} f_1(xy) &= f_2(h(xy)) = \frac{f_1(x)g_2(h(y)) + f_1(y)g_2(h(x))}{2}, \\ f_1(xy) &= \frac{f_1(x)g_1(y) + f_1(y)g_1(x)}{2}. \end{aligned}$$

Consequently,

$$f_1(x)(g_2(h(y)) - g_1(y)) + f_1(y)(g_2(h(x)) - g_1(x)) = 0. \quad (3.14)$$

If $f_1(x) \neq 0$ then from (3.13) we get $g_2(h(x)) = g_1(x)$. If $f_1(x) = 0$ we choose y such that $f_1(y) \neq 0$ then from (3.14) we get $g_2(h(x)) = g_1(x)$. \square

Theorem 3.22. *Let $(\mathbf{A}_1, (f_1, g_1))$ and $(\mathbf{A}_2, (f_2, g_2))$ be dibaric algebras with spaces of invariant linear forms $J_1 = J_{f_1}$ and $J_2 = J_{f_2}$ respectively. Assume $h: \mathbf{A}_1 \rightarrow \mathbf{A}_2$ is a homomorphism.*

- (i) *h^* transforms invariant forms to invariant forms, i.e. F invariant on \mathbf{A}_2 implies h^*F is invariant on \mathbf{A}_1 :*

$$h^*(J_2) \subset J_1. \quad (3.15)$$

- (ii) If $\text{Im } h \supset \mathbf{A}_2^2$ then h^* is injective on J_2 .
- (iii) If h is injective, then the subspace $(h^*)^{-1}(J_1)$ of \mathbf{A}_2^* —which contains J_2 by (3.15)—is equal to J_2 . Moreover, F is invariant on \mathbf{A}_2 if and only if h^*F is invariant on \mathbf{A}_1 .
- (iv) If h is surjective and $\ker h \subset J_1^\perp$, then h^* maps J_2 bijectively onto J_1 .

Proof. The prove is similar to the proof of [13, Theorem 3.3.12]. \square

Theorem 3.23.

- (1) For a dibaric algebra $(\mathbf{A}, (f, g))$, let L be a linear subspace of J_f . Then $\mathbf{A}_1 = L^\perp$ is a subalgebra.
- (2) If $(\mathbf{A}, (f, g))$ is a conservative algebra then a linear form F on \mathbf{A} is invariant if and only if its restriction F_1 to \mathbf{A}_1 is invariant on \mathbf{A}_1 .
- (3) If J_1 is the space of linear forms on \mathbf{A}_1 then

$$\dim J_1 = \dim J_f - \dim L. \quad (3.16)$$

Proof. (1) If $x, y \in \mathbf{A}_1$ then $F(x) = F(y) = 0$ for any $F \in L$. Hence $F(xy) = 0$ for any $F \in L$, i.e. $xy \in \mathbf{A}_1$.

(2) If $F_1 = F|_{\mathbf{A}_1}$, where $F \in J_f$ then $F(J_f^\perp) = 0$, i.e. $F(x^2) = f(x)F(x)$, $\forall x \in \mathbf{A}$. Since $\mathbf{A}_1 \subset \mathbf{A}$ we get $F_1 \in J_1$. Conversely, if F_1 is an invariant on \mathbf{A}_1 then $f_1(J_1^\perp) = 0$. We have $\text{ann } \mathbf{A} \subseteq \text{ann } \mathbf{A}_1 \subseteq J_1^\perp$. Hence $F_1(\text{ann } \mathbf{A}_1) = 0$ and consequently $F_1(\text{ann } \mathbf{A}) = 0$. Since \mathbf{A} is conservative, we get $F_1(J_f^\perp) = 0$, i.e. $F \in J_f$.

(3) Let $i: \mathbf{A}_1 \hookrightarrow \mathbf{A}$ be embedding then $i^*: \mathbf{A}^* \hookrightarrow \mathbf{A}_1^*$. Hence $i^*: J_f \rightarrow J_1$, $\ker i^* = L$. By (2) for any $F_1 \in J_1$ there exists $F \in J_f$ such that $F_1 = F|_{\mathbf{A}_1}$, consequently i^* is surjective. Therefore we get (3.16). \square

4. ALGEBRA OF A BISEXUAL POPULATION

In this section we consider the evolution algebra \mathcal{B} of a bisexual population.

For $z = (x, y) = \sum_{i=1}^n x_i e_i^{(f)} + \sum_{j=1}^\nu y_j e_j^{(m)} \in \mathcal{B}$ denote

$$\begin{aligned} \tilde{\mathcal{B}}^* &= \left\{ (f, g) \in \mathcal{B}^* \times \mathcal{B}^* : f(z^2) = g(z^2) = f(z)g(z) \right\}, \\ \tilde{\mathcal{B}}_{01}^* &= \left\{ (0, g) \in \mathcal{B}^* \times \mathcal{B}^* : g(z) = \sum_{i=1}^n \gamma_i x_i + \sum_{j=1}^\nu \delta_j y_j, \text{ with} \right. \\ &\quad \left. \sum_{k=1}^n P_{ij,k}^{(f)} \gamma_k + \sum_{l=1}^\nu P_{ij,l}^{(m)} \delta_l = 0, \forall i, j \right\}, \\ \tilde{\mathcal{B}}_{10}^* &= \left\{ (f, 0) \in \mathcal{B}^* \times \mathcal{B}^* : f(z) = \sum_{i=1}^n \alpha_i x_i + \sum_{j=1}^\nu \beta_j y_j, \text{ with} \right. \\ &\quad \left. \sum_{k=1}^n P_{ij,k}^{(f)} \alpha_k + \sum_{l=1}^\nu P_{ij,l}^{(m)} \beta_l = 0, \forall i, j \right\}, \end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{B}}_{12}^* &= \left\{ (f, g) \in \mathcal{B}^* \times \mathcal{B}^* : f(z) = \sum_{i=1}^n \alpha_i x_i, g(z) = \sum_{j=1}^{\nu} \delta_j y_j, \text{ with} \right. \\ &\quad \left. \sum_{k=1}^n P_{ij,k}^{(f)} \alpha_k = \sum_{l=1}^{\nu} P_{ij,l}^{(m)} \delta_l = \alpha_i \delta_j, \forall i, j \right\}, \\ \tilde{\mathcal{B}}_{21}^* &= \left\{ (f, g) \in \mathcal{B}^* \times \mathcal{B}^* : f(z) = \sum_{i=1}^{\nu} \beta_i y_i, g(z) = \sum_{j=1}^n \gamma_j x_j, \text{ with} \right. \\ &\quad \left. \sum_{k=1}^n P_{ij,k}^{(f)} \gamma_k = \sum_{l=1}^{\nu} P_{ij,l}^{(m)} \beta_l = \gamma_i \beta_j, \forall i, j \right\}.\end{aligned}$$

The following theorem describes the set $\tilde{\mathcal{B}}^*$.

Theorem 4.1. *The set $\tilde{\mathcal{B}}^*$ has the form*

$$\tilde{\mathcal{B}}^* = \{(0, 0)\} \cup \tilde{\mathcal{B}}_{01}^* \cup \tilde{\mathcal{B}}_{10}^* \cup \tilde{\mathcal{B}}_{12}^* \cup \tilde{\mathcal{B}}_{21}^*.$$

Proof. Let

$$f(z) = \sum_{i=1}^n \alpha_i x_i + \sum_{j=1}^{\nu} \beta_j y_j, g(z) = \sum_{i=1}^n \gamma_i x_i + \sum_{j=1}^{\nu} \delta_j y_j.$$

From $f(z^2) = g(z^2)$ we obtain

$$\sum_{k=1}^n P_{ij,k}^{(f)} (\alpha_k - \gamma_k) + \sum_{l=1}^{\nu} P_{ij,l}^{(m)} (\beta_l - \delta_l) = 0, \forall i, j.$$

From $f(z^2) = f(z)g(z)$ we get

$$\alpha_i \gamma_j = 0, \forall i, j = 1, \dots, n; \quad \beta_i \delta_j = 0, \forall i, j = 1, \dots, \nu; \quad (4.1)$$

$$\sum_{k=1}^n P_{ij,k}^{(f)} \alpha_k + \sum_{l=1}^{\nu} P_{ij,l}^{(m)} \beta_l = \alpha_i \delta_j + \gamma_i \beta_j, \text{ for all } i = 1, \dots, n; j = 1, \dots, \nu.$$

By (4.1) it is easy to see that if there exists i_0 such that $\alpha_{i_0} \neq 0$ then $\gamma_j = 0$ for any $j = 1, \dots, n$. Similarly, if there exists i_1 such that $\delta_{i_1} \neq 0$ then $\beta_j = 0$ for all $j = 1, \dots, \nu$. Therefore if f depends only on x (resp. y) then g depends only on y (resp. x). Moreover if f (resp. g) depends on both x and y then $g = 0$ (resp. $f = 0$). \square

For $z = (x, y) \in \mathcal{B}$ denote $X(z) = \sum_{i=1}^n x_i$ and $Y(z) = \sum_{i=1}^{\nu} y_i$. It is easy to see that $(X(z), Y(z)) \in \tilde{\mathcal{B}}_{12}^*$, hence $(X(z), Y(z))$ is a bq-homomorphism.

In this section we consider only X -invariant linear forms, the Y -linear forms can be obtained from X -linear forms by replacing n with ν and x with y .

Denote by $J = J_X$ the set of all X -invariant linear forms of \mathcal{B} . The set J is a subspace of the dual space \mathcal{B}^* .

Denote $\mathbb{P}_i^{(m)} = \left(P_{ij,k}^{(m)} \right)_{j,k=1,\dots,\nu}$, $i = 1, \dots, n$.

Proposition 4.2.

- (1) $\det(\mathbb{P}_i^{(m)} - I) = 0$ for any $i = 1, \dots, n$, where I is the $(\nu \times \nu)$ -identity matrix.
 (2) For any $d \in \{1, \dots, \nu\}$ there is $\mathbb{P}_i^{(m)}$ such that $\dim J = d$.

Proof. (1) From (2.1) it follows that sum of all columns of $\mathbb{P}_i^{(m)} - I$ is equal to zero for any i . Consequently, the columns are linearly dependent.

(2) Take $F(z) = \sum_{i=1}^n \beta_i x_i + \sum_{j=1}^\nu \alpha_j y_j$. Then from (3.2) by (2.5) we get $\beta_1 = \dots = \beta_n = 0$ and

$$\sum_{l=1}^\nu P_{ij,l}^{(m)} \alpha_l = \alpha_j, \quad \text{for all } i = 1, \dots, n; j = 1, \dots, \nu. \quad (4.2)$$

Using (1) we get that the system (4.2) has a set $W_i \neq \{0\}$ of solutions for any $i = 1, \dots, n$. Moreover one can see that $\alpha_1 = \dots = \alpha_\nu = \text{constant}$ is a solution of the system for any i . Therefore $W = \cap_{i=1}^n W_i \neq 0$. Consequently,

$$J = \left\{ f(z) = \sum_{j=1}^\nu \alpha_j y_j : \alpha = (\alpha_1, \dots, \alpha_\nu) \in W \right\}, \quad \dim J = \dim W.$$

For a given $d = 1, \dots, \nu$ one can choose $\mathbb{P}_i^{(m)}$ such that $\dim J = d$. □

From this proposition we get

Corollary 4.3. Any X -invariant (resp. Y -invariant) linear form $F(z) = F(x, y)$ depends only on y (resp. on x).

Let us consider an example.

Example 1. Consider the case $n = 1, \nu = 2$. In this case $P_{11,1}^{(f)} = P_{12,1}^{(f)} = 1$. Denote $P_{11,1}^{(m)} = a, P_{11,2}^{(m)} = 1 - a, P_{12,1}^{(m)} = b, P_{12,2}^{(m)} = 1 - b$. In this case the system (4.2) gets the following form

$$(a - 1)(\alpha_1 - \alpha_2) = 0, \quad b(\alpha_1 - \alpha_2) = 0.$$

If $a = 1, b = 0$ then any $(\alpha_1, \alpha_2), \alpha_1, \alpha_2 \in \mathbb{R}$, is a solution to the system; if $1 - a + b \neq 0$ then $(\alpha_1, \alpha_1), \alpha_1 \in \mathbb{R}$, is the only solution. Therefore

$$\dim J = \begin{cases} 2, & \text{if } a = 1, b = 0; \\ 1, & \text{otherwise.} \end{cases}$$

The following example shows that the invariant linear forms on $S^{n-1} \times S^{\nu-1}$ may depend on both x and y .

Example 2. Consider the case $n = 3, \nu = 2$. Consider the evolution operator $V: S^3 \times S^2 \rightarrow S^3 \times S^2$, $z = (x_1, x_2, x_3, y_1, y_2) \mapsto z' = (x'_1, x'_2, x'_3, y'_1, y'_2)$ given by:

$$\begin{aligned} x'_1 &= x_1 y_1 + \frac{1}{2} x_3 y_1 \\ x'_2 &= x_2 y_2 + \frac{1}{2} x_3 y_2 \\ x'_3 &= x_1 y_2 + x_2 y_1 + \frac{1}{2} x_3 y_1 + \frac{1}{2} x_3 y_2 \\ y'_1 &= x_1 y_1 + x_1 y_2 + \frac{1}{2} x_3 y_1 + \frac{1}{2} x_3 y_2 \\ y'_2 &= x_2 y_1 + x_2 y_2 + \frac{1}{2} x_3 y_1 + \frac{1}{2} x_3 y_2. \end{aligned} \tag{4.3}$$

For the algebra \mathcal{B} corresponding to the operator (4.3) we have the following invariant linear forms:

$$f_1(z) = \frac{1}{3}(2x_1 + x_3 + y_1), \quad f_2(z) = \frac{1}{3}(2x_2 + x_3 + y_2).$$

Lemma 4.4.

(1) $z = (x, y) \in \text{ann } \mathcal{B}$ if and only if

$$\begin{aligned} \sum_{i=1}^n P_{ij,k}^{(f)} x_i &= 0, \quad k = 1, \dots, n; & \sum_{i=1}^n P_{ij,l}^{(m)} x_i &= 0, \quad l = 1, \dots, \nu, \quad \text{for all } j; \\ \sum_{j=1}^{\nu} P_{ij,k}^{(f)} y_j &= 0, \quad k = 1, \dots, n; & \sum_{j=1}^{\nu} P_{ij,l}^{(m)} y_j &= 0, \quad l = 1, \dots, \nu, \quad \text{for all } i. \end{aligned} \tag{4.4}$$

(2) If $z \in \text{ann } \mathcal{B}$ then $X(z) = Y(z) = 0$.

Proof. (1) Let $z = (x, y), t = (u, v) \in \mathbb{R}^{n+\nu}$ with $z \in \text{ann } \mathcal{B}$. Using (2.4) from $zt = 0$ we get

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^{\nu} P_{ij,k}^{(f)} (x_i v_j + u_i y_j) &= 0, \quad k = 1, \dots, n, \\ \sum_{i=1}^n \sum_{j=1}^{\nu} P_{ij,l}^{(m)} (x_i v_j + u_i y_j) &= 0, \quad l = 1, \dots, \nu. \end{aligned} \tag{4.5}$$

Since (4.5) must be true for any t , we take $t = (u, v)$ with $u_1 = \dots = u_n = 0, v_{j_0} = 1$, for some j_0 and $v_j = 0, j \neq j_0$. Then we get

$$\sum_{i=1}^n P_{ij_0,k}^{(f)} x_i = 0, \quad \sum_{i=1}^n P_{ij_0,l}^{(m)} x_i = 0.$$

By arbitrariness of j_0 we get the first line of the condition (4.4). The second line can be obtained similarly.

Now assume that z satisfies the condition (4.4). Then from (2.4) it easily follows that $zt = 0$ for any $t \in \mathcal{B}$, i.e. $z \in \text{ann } \mathcal{B}$.

(2) From equations (4.4) we get

$$\sum_{k=1}^n \left(\sum_{i=1}^n P_{ij,k}^{(f)} x_i \right) = 0,$$

i.e.

$$\sum_{i=1}^n \left(\sum_{k=1}^n P_{ij,k}^{(f)} \right) x_i = \sum_{i=1}^n x_i = X(z) = 0.$$

Equality $Y(z) = 0$ can be obtained by a similar way. \square

Proposition 4.5. *An EABP \mathcal{B} is an algebra induced by a linear operator if and only if*

$$P_{ij,k}^{(f)} = P_{1j,k}^{(f)}, \quad P_{ij,l}^{(m)} = P_{1j,l}^{(m)} \quad \text{for all } i, k = 1, \dots, n; \quad j, l = 1, \dots, \nu. \quad (4.6)$$

Moreover the linear operator A has the form $A(z) = 2ze_1^{(f)}$.

Proof. From equation (3.10) for $z = t = e_i^{(f)}$, we get $A(e_i^{(f)}) = 0$, $i = 1, \dots, n$. For $z = (x, y), t = (u, v) \in \mathcal{B}$ from (3.10) we get

$$\begin{aligned} & 2 \left(\sum_{i=1}^n x_i e_i^{(f)} + \sum_{j=1}^{\nu} y_j e_j^{(m)} \right) \left(\sum_{i=1}^n u_i e_i^{(f)} + \sum_{j=1}^{\nu} v_j e_j^{(m)} \right) = 2 \sum_{i=1}^n \sum_{j=1}^{\nu} (x_i v_j + u_i y_j) e_i^{(f)} e_j^{(m)} \\ & = \left(\sum_{i=1}^n u_i \left(\sum_{i=1}^n x_i A(e_i^{(f)}) + \sum_{j=1}^{\nu} y_j A(e_j^{(m)}) \right) \right) + \left(\sum_{i=1}^n x_i \left(\sum_{i=1}^n u_i A(e_i^{(f)}) + \sum_{j=1}^{\nu} v_j A(e_j^{(m)}) \right) \right). \end{aligned}$$

Consequently,

$$\begin{aligned} & 2 \sum_{i=1}^n \sum_{j=1}^{\nu} (x_i v_j + u_i y_j) e_i^{(f)} e_j^{(m)} = \sum_{i=1}^n u_i \sum_{j=1}^{\nu} y_j A(e_j^{(m)}) + \sum_{i=1}^n x_i \sum_{j=1}^{\nu} v_j A(e_j^{(m)}) \\ & = \sum_{i=1}^n \sum_{j=1}^{\nu} (x_i v_j + u_i y_j) A(e_j^{(m)}). \end{aligned}$$

Therefore, the last equality is true iff $2e_i^{(f)} e_j^{(m)} = A(e_j^{(m)})$ for any $i = 1, \dots, n; \quad j = 1, \dots, \nu$. This equality is satisfied iff $e_i^{(f)} e_j^{(m)} = e_1^{(f)} e_j^{(m)}$ for any $i = 1, \dots, n; \quad j = 1, \dots, \nu$ which is equivalent to the condition (4.6). Using $A(e_i^{(f)}) = 0$ and $A(e_j^{(m)}) = 2e_1^{(f)} e_j^{(m)}$ one gets that $A(z) = 2ze_1^{(f)}$. \square

Remark 4.6. *In the class of baric algebras the algebra induced by a linear operator A is Bernstein if and only if $A^2 = A$, i.e. A is a projection, and in this case the algebra is necessarily conservative. But in our (non-baric) case this property is not true. Indeed, for $z = (x, y) \in \mathcal{B}$ we get $X(z) = \sum_{i=1}^n x_i$, $X(A(z)) = X(2ze_1^{(f)}) = \sum_{j=1}^{\nu} y_j$, i.e. $X(z) \neq X(A(z))$ in general.*

Given two EABP algebras $\mathcal{B}_1, \mathcal{B}_2$ a homomorphism $h: \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is a linear mapping with $h(zt) = h(z)h(t)$ and $X(h(z)) = X(z)$.

Theorem 4.7. *Let \mathcal{B}_1 and \mathcal{B}_2 be EABPs. If $h: \mathcal{B}_1 \rightarrow \mathcal{B}_2$ is a homomorphism then*

$$\begin{aligned} X(h(e_i^{(f)})) &= 1, & X(h(e_j^{(m)})) &= 0, \\ Y(h(e_i^{(f)})) &= 0, & Y(h(e_j^{(m)})) &= 1, \end{aligned} \quad i = 1, \dots, n; \quad j = 1, \dots, \nu.$$

Proof. The first two equalities easily follow from $X(z) = X(h(z))$. Now we shall prove the third and fourth equalities. Assume h on basis elements is given as follows

$$\begin{aligned} h(e_i^{(f)}) &= \sum_{j=1}^n \alpha_{ij} e_j^{(f)} + \sum_{k=1}^{\nu} \beta_{ik} e_k^{(m)}, \quad i = 1, \dots, n, \\ h(e_j^{(m)}) &= \sum_{i=1}^n \lambda_{ji} e_i^{(f)} + \sum_{l=1}^{\nu} \mu_{jl} e_l^{(m)}, \quad j = 1, \dots, \nu. \end{aligned}$$

We have $X(h(e_i^{(f)})) = \sum_{j=1}^n \alpha_{ij} = 1$, $X(h(e_j^{(m)})) = \sum_{i=1}^n \lambda_{ji} = 0$. From $h(e_i^{(f)} e_i^{(f)}) = 0$ we get

$$\sum_{j=1}^n \sum_{k=1}^{\nu} \alpha_{ij} \beta_{ik} P_{jk,l}^{(f)} = 0, \quad l = 1, \dots, n, \quad \sum_{j=1}^n \sum_{k=1}^{\nu} \alpha_{ij} \beta_{ik} P_{jk,q}^{(m)} = 0, \quad q = 1, \dots, \nu. \quad (4.7)$$

From the first equality of (4.7) we get

$$\sum_{l=1}^n \left(\sum_{j=1}^n \sum_{k=1}^{\nu} \alpha_{ij} \beta_{ik} P_{jk,l}^{(f)} \right) = \sum_{j=1}^n \alpha_{ij} \sum_{k=1}^{\nu} \beta_{ik} = \sum_{k=1}^{\nu} \beta_{ik} = Y(e_i^{(f)}) = 0.$$

It is easy to check that the condition $h(e_i^{(f)} e_s^{(m)}) = h(e_i^{(f)}) h(e_s^{(m)})$ is equivalent to the following equations

$$\begin{aligned} \sum_{j=1}^n \sum_{l=1}^{\nu} \alpha_{ij} \mu_{sl} P_{jl,q}^{(f)} + \sum_{t=1}^n \sum_{k=1}^{\nu} \lambda_{st} \beta_{ik} P_{tk,q}^{(f)} &= \sum_{j=1}^n \alpha_{jq} P_{is,j}^{(f)} + \sum_{l=1}^{\nu} \lambda_{lq} P_{is,l}^{(m)}, \\ \sum_{j=1}^n \sum_{l=1}^{\nu} \alpha_{ij} \mu_{sl} P_{jl,\eta}^{(m)} + \sum_{t=1}^n \sum_{k=1}^{\nu} \lambda_{st} \beta_{ik} P_{tk,\eta}^{(m)} &= \sum_{j=1}^n \beta_{j\eta} P_{is,j}^{(f)} + \sum_{l=1}^{\nu} \mu_{l\eta} P_{is,l}^{(m)}. \end{aligned} \quad (4.8)$$

Summing the equation (4.8) by $q = 1, \dots, n$ and using above obtained relations we get $Y(e_s^{(m)}) = \sum_{l=1}^{\nu} \mu_{sl} = 1$, $s = 1, \dots, \nu$. \square

Let \mathcal{B} be an EABP algebra with basis set $e_i^{(f)}$, $i = 1, \dots, n$; $e_j^{(m)}$, $j = 1, \dots, \nu$. We say $e_i^{(f)}$ (resp. $e_j^{(m)}$) occurs in $z \in \mathcal{B}$, if the coefficient α_i (resp. β_j) is nonzero in $z = \sum_i \alpha_i e_i^{(f)} + \sum_j \beta_j e_j^{(m)}$.

The following example shows that in $h(e_i^{(f)})$ may occur $e_j^{(f)}$ for some j and $e_k^{(m)}$ for some k .

Example 3. *Consider the EABP of Example 1, i.e. case $n = 1$, $\nu = 2$. In this case*

$$e_1^{(f)} e_1^{(m)} = e_1^{(f)} + a e_1^{(m)} + (1-a) e_2^{(m)}, \quad e_1^{(f)} e_2^{(m)} = e_1^{(f)} + b e_1^{(m)} + (1-b) e_2^{(m)}.$$

One can see that $h(e_1^{(f)}) = e_1^{(f)} + \alpha e_1^{(m)} - \alpha e_2^{(m)}$, and $\alpha \neq 0$ if $a = b$.

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REFERENCES

- [1] R. Andrade, A. Labra, On a class of baric algebras, *Linear Algebra Appl.* 245 (1996), 49–53.
- [2] R. Baeza-Vega, R. Benavides, Associative bilinear forms in some baric algebras, *Linear Algebra Appl.* 180 (1993), 219–226.
- [3] J. Bernad, S. González, C. Martínez, A.V. Il'tyakov, On identities of baric algebras and superalgebras, *J. Algebra* 197(2)(1997), 385–408.
- [4] R. Costa, H.Jr. Guzzo, Indecomposable baric algebras I, II, *Linear Algebra Appl.* 183 (1993), 223–236, 196 (1994), 233–242.
- [5] M. A. Couto, J.C. Gutiérrez Fernández, Dibaric algebras, *Proyecciones* 19(3) (2000), 249–269.
- [6] I.M.H. Etherington, Genetic algebras, *Proc. Roy. Soc. Edinburgh* 59 (1939), 242–258.
- [7] I.M.H. Etherington, Duplication of linear algebras, *Proc. Edinburgh Math. Soc. (2)* 6 (1941), 222–230.
- [8] I.M.H. Etherington, Non-associative algebra and the symbolism of genetics, *Proc. Roy. Soc. Edinburgh. Sect. B* 61 (1941), 24–42.
- [9] J.C.M. Ferreira, H. Jr. Guzzo, On the bar-radical of Jordan baric algebras, *Results Math.* 51 (2007), 43–49.
- [10] H. Jr. Guzzo, Some properties of bar-radical of baric algebras, *Comm. Algebra* 30 (2002), 4827–4835.
- [11] P. Holgate, Genetic algebras associated with sex linkage, *Proc. Edinburgh Math. Soc. (2)* 17 (1970/71), 113–120.
- [12] M. Ladra, U.A. Rozikov, Evolution algebra of a bisexual population, arXiv:1003.2541v1, 2010.
- [13] Y.I. Lyubich, *Mathematical structures in population genetics*, Springer-Verlag, Berlin, 1992.
- [14] L.A. Peresi, On derivations of baric algebras with prescribed automorphisms, *Linear Algebra Appl.* 104 (1988), 71–74.
- [15] M.L. Reed, Algebraic structure of genetic inheritance, *Bull. Amer. Math. Soc. (N.S.)* 34(2) (1997), 107–130.
- [16] J. P. Tian, *Evolution algebras and their applications*, Lecture Notes in Mathematics, 1921, Springer-Verlag, Berlin, 2008.
- [17] A. Wörz-Busekros, *Algebras in genetics*, Lecture Notes in Biomathematics, 36, Springer-Verlag, Berlin-New York, 1980.

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